LINEAR ALGEBRA — PRACTICE EXAM 1

(1) **Matrix algebra.**

Let $A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 3 & 0 \\ 2 & -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 & 0 \\ -6 & 0 & 1 & 1 \\ 0 & 5 & 10 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \\ -1 & -2 & -3 \end{bmatrix}$. Compute $(A+C)^T B$.

ANSWER:

$$A + C = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 3 & 0 \\ 2 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 6 & 4 \\ 1 & -3 & -5 \end{bmatrix}$$

$$(A+C)^T B = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 6 & -3 \\ 0 & 4 & -5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 0 \\ -6 & 0 & 1 & 1 \\ 0 & 5 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 19 & 1 \\ -33 & -16 & -22 & 6 \\ -24 & -25 & -46 & 4 \end{bmatrix}$$

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(2) **Systems of linear equations.**

Give a parametric solution to the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 &= -4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 - 9x_5 &= -4 \\ 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 &= -4 \end{aligned}$$

If there are solutions, verify that at least one of your solutions is correct.

ANSWER:

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 & -4 \\ 3 & 6 & -2 & 1 & -9 & -4 \\ 2 & 4 & -2 & 2 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 & 0 & -4 \\ 0 & 0 & 1 & -5 & -9 & 8 \\ 2 & 4 & -2 & 2 & 4 & -4 \end{bmatrix} \qquad \text{II} - 3\text{I} \to \text{II}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 0 & -4 \\ 0 & 0 & 1 & -5 & -9 & 8 \\ 0 & 0 & 0 & -2 & 4 & 4 \end{bmatrix} \qquad \text{III} - 2\text{I} \to \text{III}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 0 & -4 \\ 0 & 0 & 1 & -5 & -9 & 8 \\ 0 & 0 & 0 & 1 & -2 & -2 \end{bmatrix} \qquad -\frac{1}{2}\text{III} \to \text{III}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 0 & -19 & -2 \\ 0 & 0 & 0 & 1 & -2 & -2 \end{bmatrix} \qquad \text{II} + 5\text{III} \to \text{II}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & -19 & -2 \\ 0 & 0 & 0 & 1 & -2 & -2 \end{bmatrix} \qquad \text{I} - 2\text{III} \to \text{I}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & -15 & -2 \\ 0 & 0 & 1 & 0 & -19 & -2 \\ 0 & 0 & 0 & 1 & -2 & -2 \end{bmatrix} \qquad \text{I} + \text{II} \to \text{I}$$

This last matrix corresponds to the system of equations:

$$x_1 + 2x_2 - 15x_5 = -2$$

$$\begin{aligned} x_2 &= x_2 \\ x_3 - 19x_5 &= -2 \\ x_4 - 2x_5 &= -2 \\ x_5 &= x_5 \end{aligned}$$

In parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 15 \\ 0 \\ 19 \\ 2 \\ 1 \end{bmatrix}$$

We verify the solution by showing that $A\vec{x} = \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}$, where A is the coefficient matrix for the original system of equations.

$$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 3 & 6 & -2 & 1 & -9 \\ 2 & 4 & -2 & 2 & 4 \end{bmatrix} \left(\begin{bmatrix} -2 \\ 0 \\ -2 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 15 \\ 0 \\ 19 \\ 2 \\ 1 \end{bmatrix} \right).$$

It is easier to distribute and do each of the matrix multiplications separately.

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 3 & 6 & -2 & 1 & -9 \\ 2 & 4 & -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+2-4 \\ -6+4-2 \\ -4+4-4 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}$$

$$x_2 \begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 3 & 6 & -2 & 1 & -9 \\ 2 & 4 & -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2+2 \\ -6+6 \\ -4+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_5 \begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 3 & 6 & -2 & 1 & -9 \\ 2 & 4 & -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 15 \\ 0 \\ 19 \\ 2 \\ 1 \end{bmatrix} = x_5 \begin{bmatrix} 15-19+4 \\ 45-38+2-9 \\ 30-38+4+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus we have verified our answer.

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(3) **Linear independence**

Determine whether the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -6 \\ -3 \end{bmatrix} \right\}$$

is linearly independent or linearly dependent. If the set of vectors is linearly dependent, give one nontrivial relation between the vectors, and verify that it is indeed a nontrivial relation.

ANSWER:

$$\begin{bmatrix} 2 & 4 & 1 & -2 \\ -1 & 0 & -2 & 3 \\ 0 & -3 & 3 & -6 \\ 3 & 1 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -2 & 3 \\ 0 & -3 & 3 & -6 \\ 2 & 4 & 1 & -2 \\ 3 & 1 & 4 & -3 \end{bmatrix} \qquad \text{swap rows around}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 2 & 4 & 1 & -2 \\ 3 & 1 & 4 & -3 \end{bmatrix} \qquad -\mathrm{I} \to \mathrm{I}, -\frac{1}{3}\mathrm{II} \to \mathrm{II}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 4 & -3 & 4 \\ 0 & 1 & -2 & 6 \end{bmatrix} \qquad \mathrm{III} - 2\mathrm{I} \to \mathrm{III}, \mathrm{IV} - 3\mathrm{I} \to \mathrm{IV}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{bmatrix} \qquad \mathrm{III} - 4\mathrm{II} \to \mathrm{III}, \mathrm{IV} - \mathrm{I} \to \mathrm{IV}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{I} - 2\mathrm{III} \to \mathrm{I}, \mathrm{II} + \mathrm{III} \to \mathrm{II}, \mathrm{IV} + \mathrm{III} \to \mathrm{IV}$$

The vectors are not linearly independent because the matrix does not have a pivot in every column. A non-trivial relationship between the vectors is given by the coefficients in the non-pivot column: $5\mathbf{v}_1 - 2\mathbf{v}_2 - 4\mathbf{v}_3 = \mathbf{v}_4$. We verify that is the case:

$$5\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10-8-4 \\ -5+8 \\ 6-12 \\ 15-2-16 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -6 \\ -3 \end{bmatrix}.$$

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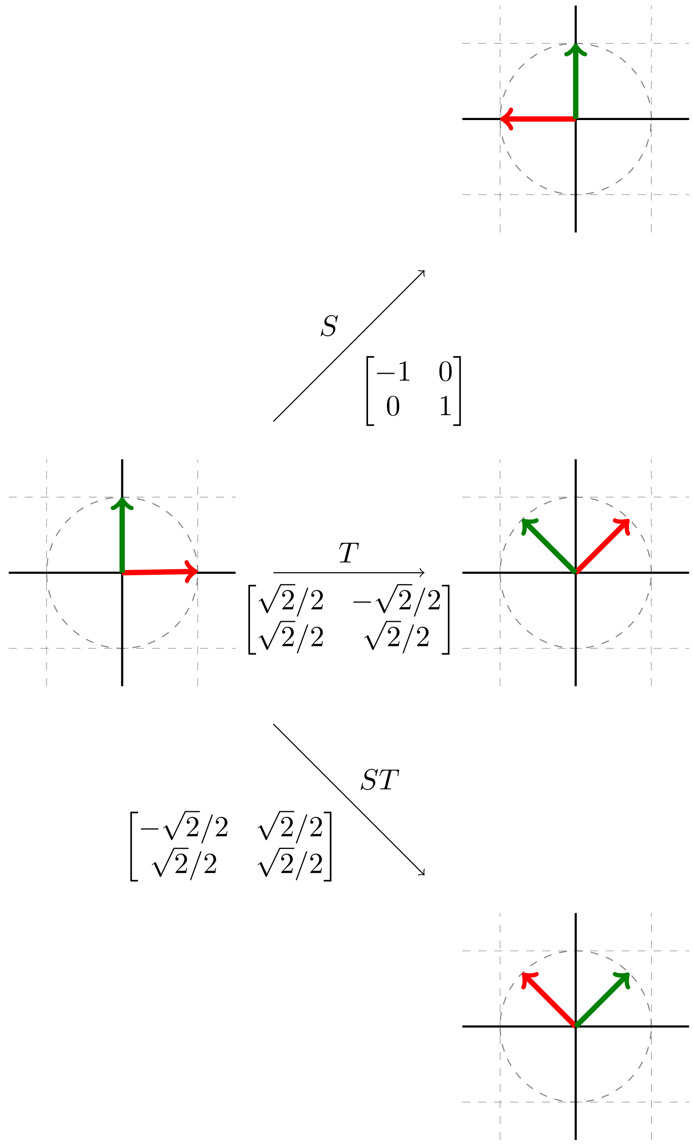
(4) **Linear transformations of the plane**

Let S and T be linear transformations of the plane, where S is reflection over the y-axis, and T is rotation by 45° ($\pi/4$ radians) about the origin in the counter-clockwise direction. Give the matrices A, B, and C associated to the linear transformations S, T, and ST respectively. That is, find the matrices A, B, and C satisfying $S(\mathbf{x}) = A\mathbf{x}$, $T(\mathbf{x}) = B\mathbf{x}$, and $(ST)(\mathbf{x}) = C\mathbf{x}$.

For each map, draw a picture showing how the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are transformed under S, T, and ST. Label your vectors clearly.

ANSWER:


S
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
T
$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$
ST
$\begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$